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L^p ESTIMATES FOR SCHRÖDINGER EVOLUTION EQUATIONS

BY

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ABSTRACT. We prove that for Cauchy data in $L^1(\mathbf{R}^n)$, the solution of a Schrödinger evolution equation with constant coefficients of order $2m$ is uniformly bounded for $t \neq 0$, with bound $(1 + |t|^{-c})$, where c is an integer, $c > n/2m - 1$. Moreover it belongs to $L^q(\mathbf{R}^n)$ if $q > q(m, n)$, with its L^q norm bounded by $(|t|^{c'} + |t|^{-c})$, where c' is an integer, $c' > n/q$. A maximal local decay result is proved. Interpolating between L^1 and L^2, we derive (L^p, L^q) estimates.

On the other hand, we prove that for Cauchy data in $L^p(\mathbf{R}^n)$, such a Cauchy problem is well posed as a distribution in the t-variable with values in $L^p(\mathbf{R}^n)$, and we compute the order of the distribution. We apply these two results to the study of Schrödinger equations with potential in $L^p(\mathbf{R}^n)$. We give an estimate of the resolvent operator in that case, and prove an asymptotic boundedness for the solution when the Cauchy data belongs to a subspace of $L^p(\mathbf{R}^n)$.

1. Introduction. In this paper we study Schrödinger evolution equations

$$(*) \qquad \partial U/\partial t = (iP(D) + V(x))U, \quad U(0, x) = U_0(x) \in L^p(\mathbf{R}^n),$$

where

$$D = \left(\frac{1}{i}\frac{\partial}{\partial x_1}, \ldots, \frac{1}{i}\frac{\partial}{\partial x_n}\right)$$

and $P(\xi)$ is an elliptic polynomial of order $2m$ with $\operatorname{Im} P_{2m}(\xi) = 0$. It is well known that in the case where $\operatorname{Im} P_{2m}(\xi) > 0$ (Heat equation), the solution belongs to $L^p(\mathbf{R}^n)$ in the x-variables for $t \geq 0$. In the case where $\operatorname{Im} P_{2m}(\xi) < 0$ (backward Heat equation), there are no L^p estimates of the solution by the L^p norm of the Cauchy data. We are concerned with the limiting case of the Schrödinger equation where $\operatorname{Im} P_{2m}(\xi) = 0$.

Hörmander [**7**] proved that even in the simplest case (where P is the Laplacian of $\mathbf{R}^n$) the problem is not well posed in the usual sense in $L^p(\mathbf{R}^n)$ if $p \neq 2$: $e^{-it\xi^2}$ is not a multiplier of L^p $(p \neq 2)$. This implies that the Hille-Yosida estimates of the resolvent of $(iP(D) + V(x))$ viewed as an operator in $L^p(\mathbf{R}^n)$ are not fulfilled. In Balabane and Emami-Rad [**1**] we proved that for the usual Schrödinger equation $(P(D) = \Delta, V = 0)$ the problem $(*)$ is well posed in $L^p(\mathbf{R}^n)$, for any p, in the distribution sense in the t-variable. In [**2**] we proved that the result remains true with $V = 0$ and $P(D)$ a homogeneous system with constant coefficients. For that

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aim, the abstract tool of Smooth Distribution Semigroups was introduced and a Hille-Yosida theorem proved for such semigroups.

In this paper we study the equation including a potential $V(x) \in L^r(\mathbf{R}^n)$ and we drop the homogeneity condition. We prove that in this case, the problem $(*)$ is still well posed in the distribution sense in the t-variable with values in $L^p(\mathbf{R}^n)$, and we compute the order of the distribution. We derive precise estimates of the $\mathcal{L}(L^p)$ norm of the resolvent operator $(\lambda I - (iP(D)+V))^{-1}$. We prove uniform boundedness for the solution when Cauchy data belongs to a subspace of $L^p(\mathbf{R}^n)$.

Another related problem is the uniform boundedness of the solution of the Cauchy problem $(*)$ when U_0 belongs to $L^1(\mathbf{R}^n)$ and for $t \neq 0$. We prove that in the constant coefficients case, if the restriction of $P(\xi)$ to the unit sphere fulfills a nondegeneracy condition, then the solution belongs to $L^\infty(\mathbf{R}^n)$ for $t \neq 0$. For $n > 3 + 2/m - 1$ it belongs to $L^q(\mathbf{R}^n)$ for $q > q(m,n)$. It decays locally as $|t|^{-n/2}$. This is done using the foliation of $\mathbf{R}^n$ by the wave surfaces $P(\xi) = cte$, and applying the stationary phase method to estimate the integrals involved. As a corollary we prove $(L^p, L^{p'})$ and (L^p, L^q) estimates for the Cauchy problem $(*)$. (Estimates of this type have been given by Brenner [4] for the wave equation using hyperbolicity.)

In §2 (L^1, L^q) estimates are proved for $V = 0$. In §3 $(L^p, L^{p'})$ estimates are derived, and (L^p, L^q) estimates are established. The Cauchy problem $(*)$ in L^p with $V = 0$ is studied in §4. Smooth distribution semigroups are introduced. In §5 the Cauchy problem $(*)$ is solved and (L^p, L^p) estimates are given for the solution and for the resolvent operator.

REMARK. As usual the notation (L^p, L^q) means estimates of the L^q norm of the solution of $(*)$ when the Cauchy data belongs to L^p. p' is the conjugate index of p.

2. Behaviour of the solutions for Cauchy data in $L^1(\mathbf{R}^n)$. We consider the Cauchy problem (with constant coefficients)

$$(**) \qquad \partial U/\partial t = iP(D)U, \quad U(0,x) = U_0(x),$$

where $U_0(x) \in S(\mathbf{R}^n)$; the solution is given by

$$U(t,x) = \overline{\mathcal{F}}(e^{itP(\xi)}) * U_0$$

(where $\mathcal{F}$ denotes the usual Fourier transform, $\overline{\mathcal{F}}$ its inverse, and $\overline{\mathcal{F}}(e^{itP(\xi)})$ is defined as an oscillatory integral).

The aim of this section is to estimate the L^q norm in the x-variable of $U(t,x)$ by the L^1 norm of U_0. So what we have to prove is that $\overline{\mathcal{F}}(e^{itP(\xi)})$ belongs to L^q for any fixed $t \neq 0$. This will be done using foliation of the exterior of a compact set of $\mathbf{R}^n$ by the wave surfaces of P. The Stationary Phase Method (Duistermaat [**5**]) then gives the behaviour of the integrals defining $\overline{\mathcal{F}}(e^{itP(\xi)})$.

Suitable hypotheses for proving the estimates are:

(H1) $P(\xi)$ is a real valued elliptic polynomial, with principal part $p(\xi)$ of degree $2m$.

(H2) For $u \in S^{n-1}$ (the unit sphere of $\mathbf{R}^n$), the restriction to S^{n-1} of $\psi(\xi) = \langle u, \xi\rangle p^{-1/2m}(\xi)$ is nondegenerate at its critical points (i.e. $d^2_{\omega\omega}(\langle u, \omega\rangle p^{-1/2m}(\omega))$ is a nondegenerate quadratic form on $T_\omega S^{n-1}$ if $\omega \in S^{n-1}$ and $d_\omega(\langle u, \omega\rangle p^{-1/2m}(\omega)) = 0$).

(H3) $m \geq 1$ and $n \geq 3$, or
(H3′) $m \geq 2$ and $n > 3 + 2/(m-1)$.

Let c and c' be integers with $c > n/2m - 1$ and $c' > n/q$.
Let

$$\frac{1}{q(m,n)} = \frac{(m-1)(n-3)}{(2m-1)n} - \frac{2}{(2m-1)n}.$$

The estimates are

THEOREM 1. (a) *If* (H1), (H2) *and* (H3) *are fulfilled, the solution* $U(t,\cdot)$ *of the Cauchy problem* (**) *with Cauchy data in* $L^1(\mathbf{R}^n)$ *belongs to* $L^\infty(\mathbf{R}^n)$ *for* $t \neq 0$. *The bound is*

$$\|U(t,\cdot)\|_{L^\infty(\mathbf{R}^n)} \leq C_\infty(1+|t|^{-c})\|U_0\|_{L^1(\mathbf{R}^n)}.$$

(b) *If* (H1), (H2) *and* (H3′) *are fulfilled, then* $U(t,\cdot)$ *belongs to* $L^q(\mathbf{R}^n)$ *for* $q(m,n) < q \leq \infty$. *The estimate is*

$$\|U(t,\cdot)\|_{L^q(\mathbf{R}^n)} \leq C_q(|t|^{c'} + |t|^{-c})\|U_0\|_{L^1(\mathbf{R}^n)}.$$

C_q *and* C_∞ *are absolute constants.*

REMARK 1. (i) If P is homogeneous, the estimates can be trivially improved to

$$\|U(t,x)\|_{L^q} \leq C_q|t|^{-n/2mq'}\|U_0\|_{L^1} \quad \text{for } q(m,n) < q \leq \infty.$$

(ii) Without any change to the bounds and to the proof, these estimates can be proved with $W^{s,q}$ norm in place of L^q norm if $q > \tilde{q}(m,n,s)$ with $\tilde{q}^{-1}(m,n,s) = q^{-1}(m,n) - s/(2m-1)n$.

(iii) Without any change to the bounds and to the proof, (H1) could be replaced by $P(\xi) = \rho^{2m}p(\omega) + Q(\rho,\omega)$ with a symbol $Q \in S^{2m-1}_{1,0}(S^{n-1} \times \mathbf{R}_+)$.

The proof of Theorem 1 will follow the lemmas below. We will assume $m \geq 2$, the case $m = 1$ can be solved by direct computation.

A. *A foliation of* $\mathbf{R}^n \backslash \{P(\xi) \leq a\}$. Let $(\rho,\omega) \in \mathbf{R}_+ \times S^{n-1}$ be the spherical coordinates in $\mathbf{R}^n$. In these variables, $P(\rho,\omega) = \rho^{2m}p(\omega) + Q(\rho,\omega)$ with degree of Q strictly less than $2m$. Ellipticity of P implies $|p(\omega)| \geq c > 0$ for $\omega \in S^{n-1}$. Since p is real valued, we can assume that $p(\omega) \geq c > 0$ for $\omega \in S^{n-1}$.

LEMMA 1. *There exist two positive constants* a *and* b, *and a function* $\rho(s,\omega) \in C^\infty(]a,\infty[\times S^{n-1})$ *such that for* $(s,\omega) \in]a,\infty[\times S^{n-1}$ *we have* $P(\rho(s,\omega),\omega) = s$ *and* $\rho(s,\omega) > b$.

PROOF. Let

$$b' = \sup_{\rho\geq 1}\sup_{\omega} |(2m\rho^{2m-2}p(\omega))^{-1}\partial Q/\partial\rho|.$$

Let $b = \max(b',1)$. For fixed $\omega \in S^{n-1}$, $P(\rho,\omega)$ is a strictly increasing function of the ρ variable for $\rho > b$, and goes to infinity when ρ does. It is then bijective from $]b,\infty[$ onto $]p(b,\omega),\infty[$. Let $\rho(s,\omega)$ be the inverse mapping, and let $a = \sup_{\omega\in S^{n-1}} P(b,\omega)$. $\rho(s,\omega)$ is then a mapping from $]a,\infty[\times S^{n-1}$ to $\mathbf{R}_+$, which verifies the identity quoted in the lemma. Moreover ρ is infinitely differentiable as the implicit function theorem asserts. Actually the mapping $(\rho,\omega) \to (s,\omega)$ is a C^∞-diffeomorphism from the open set $\mathbf{R}^n\backslash\{P(\xi) < a\}$ onto $]a,\infty[\times S^{n-1}$. Q.E.D.

Using Hörmander's definition of the symbols classes (Duistermaat [**5**]), the function just defined has the following behaviour:

LEMMA 2.

$\rho(s,\omega) = (s/p(\omega))^{1/2m} + \sigma(\omega, s)$, *where* $\sigma(\omega,s)$ *belongs to* $S^0_{1,0}(S^{n-1}\times]a,\infty[)$.

PROOF. If we let $\partial_s = \partial/\partial s$, $\partial_\omega = \partial/\partial\omega$ and $H(\omega,s) = (1+p^{-1}\rho^{-2m}Q)^{1/2m}$ we have to prove that $\rho(1-H) \in S^0_{1,0}$ and this will be done if we show that $\rho \in S^{1/2m}_{1,0}$ and $1 - H \in S^{-1/2m}_{1,0}$. First note that in the identity

$$\text{(E)} \qquad s = p\cdot\rho^{2m} + Q,$$

s goes to infinity whenever ρ does. Since the degree of Q is strictly less than $2m$, we have, uniformly in ω,

$$\text{(S1)} \qquad \rho \sim p^{-1/2m}s^{1/2m} \quad \text{as } s\to\infty.$$

Then differentiating (E) with respect to s gives

$$\text{(S2)} \qquad \partial\rho/\partial s = O(s^{-1+1/2m}).$$

By induction on α, we prove the formulas:

$$\text{(F1)} \qquad \partial_s^\alpha\rho^{2m} = 2m\rho^{2m-1}\partial_s^\alpha\rho + \sum_{i\in N_\alpha} C_i\rho^{i_0}(\partial_s\rho)^{i_1}\cdots(\partial_s^{\alpha-1}\rho)^{i_{\alpha-1}},$$

$$\text{(F2)} \qquad \partial_s^\alpha Q = \partial_\rho Q\cdot\partial_s^\alpha\rho + \sum_{j\in M_\alpha} D_j(\partial_\rho^k Q)(\partial_s\rho)^{j_1}\cdots(\partial_s^{\alpha-1}\rho)^{j_{\alpha-1}}$$

for any $\alpha\in\mathbf{N}^*$, where

$$N_\alpha = \left\{ i = (i_0,\dots,i_{\alpha-1})\in\mathbf{N}^\alpha \text{ with } \sum_{\gamma=0}^{\alpha-1} i_\gamma = 2m \text{ and } \sum_{\gamma=1}^{\alpha-1}\gamma i_\gamma = \alpha\right\},$$

$$M_\alpha = \left\{ j = (k,j_1,\dots,j_{\alpha-1})\in\mathbf{N}^\alpha \text{ with } k = \sum_{\nu=1}^{\alpha-1} j_\nu \text{ and } \sum_{\nu=1}^{\alpha-1}\nu j_\nu = \alpha\right\},$$

and C_i and D_j are absolute constants.

Then, applying ∂^α to$_s$(E) gives inductively

$$\text{(S}\alpha\text{)} \qquad \partial_s^\alpha\rho = O(s^{-\alpha+1/2m}).$$

In order to estimate s-derivatives of $(1-H)$, we note that by definition

$$\text{(T1)} \qquad 1 - H = O(s^{-1/2m}).$$

By induction on α, we prove

$$\text{(F3)} \qquad \partial_s^\alpha H = \sum_{K\in L_\alpha} E_K(\omega)H^c\rho^{-2ma-b}\prod_{\gamma=1}^{\alpha}(\partial_s^\gamma\rho)^{k_\gamma}\prod_{\nu=0}^{\alpha}(\partial_\rho^\nu Q)^{l_\nu},$$

where $E_K(\omega)\in C^\infty(S^{n-1})$ and L_α is the finite set of elements $(a,b,k,l)\in\mathbf{N}\times\mathbf{N}\times\mathbf{N}^\alpha\times\mathbf{N}^{\alpha-1}$ satisfying

$$\alpha\ge a\ge 0; \quad \alpha\ge b\ge 0; \quad \sum_1^\alpha\gamma k_\gamma = \alpha; \quad \sum_0^\alpha l_\nu = a; \quad \sum_1^\alpha k_\gamma - \sum_0^\alpha(\nu+1)l_\nu \le b-1.$$

Then using the previous estimate (Sα) on $\partial_s^\alpha \rho$ leads to

$$\text{(T}\alpha\text{)} \qquad \partial_s^\alpha H = O(s^{-\alpha-1/2m}).$$

In order to prove the same estimates for the derivatives $\partial_s^\alpha \partial_\omega^\beta$, we first prove formulas similar to (F1) and (F2) by induction on β:

$$\text{(F}'1\text{)} \qquad \partial_s^\alpha \partial_\omega^\beta \rho^{2m} = 2m\rho^{2m-1}\partial_s^\alpha \partial_\omega^\beta \rho + \sum_{i\in N_\alpha} C_i \prod_{\gamma=0}^{\alpha} \prod_{\gamma'\leq\beta} (\partial_s^\gamma \partial_\omega^{\gamma'} \rho)^{i_{\gamma,\gamma'}},$$

where $\alpha' < \beta$, $i_\gamma = \sum_{\gamma'} i_{\gamma,\gamma'}$ for $\gamma = 0, \dots, \alpha$ and

$$N_\alpha = \left\{ i = (i_0, \dots, i_\alpha) \text{ with } \sum_\gamma i_\gamma = 2m \text{ and } \sum_\gamma \gamma i_\gamma = \alpha \right\};$$

$$\text{(F}'2\text{)} \qquad \partial_s^\alpha \partial_\omega^\beta Q = \partial_\rho Q \cdot \partial_s^\alpha \partial_\omega^\beta \rho + \sum_{j\in M_\alpha} D_i(\partial_\rho^k \partial_\omega^l Q) \prod_{\nu=0}^{\alpha} \prod_{\nu'\leq\beta} (\partial_s^\nu \partial_\omega^{\nu'} \rho)^{j_{\nu,\nu'}},$$

where $k \leq \alpha$, $l \leq \beta$, $\alpha' < \beta$, $j_\nu = \sum_{\nu'} j_{\nu,\nu'}$ for $\nu = 0, \dots, \alpha$ and

$$M_\alpha = \left\{ j = (j_0, \dots, j_\alpha) \text{ with } \sum_{\nu=1}^{\alpha} j_\nu = k - j_0 \text{ and } \sum_{\nu=1}^{\alpha} \nu j_\nu = \alpha \right\}.$$

Applying $\partial_s^\alpha \partial_\omega^\beta$ to equation (E) shows that $\rho \in S_{1,0}^{1/2m}$:

$$\text{(S}'\alpha\text{)} \qquad \partial_s^\alpha \partial_\omega^\beta \rho = O(s^{-\alpha+1/2m}).$$

At last, we show inductively on β

$$\text{(F}'3\text{)} \qquad \partial_s^\alpha \partial_\omega^\beta H = \sum E_K(\omega) H^c \prod_{\gamma\leq\alpha} \prod_{\gamma'\leq\beta} (\partial_s^\gamma \partial_\omega^{\gamma'} \rho)^{k_{\gamma,\gamma'}} \prod_{\nu\leq\alpha} \prod_{\nu'\leq\beta} (\partial_\rho^\nu \partial_\omega^{\nu'} Q)^{l_{\nu,\nu'}},$$

where, if we denote by $k_\gamma = \sum_{\gamma'} k_{\gamma,\gamma'}$ and $l_\nu = \sum_{\nu'} l_{\nu,\nu'}$, then $\sum \gamma k_\gamma = \alpha$ and $\sum k_\gamma + \sum (2m - \nu - 1) l_\nu \leq -1$. This shows, using (S$\alpha$), that

$$\text{(T}'\alpha\text{)} \qquad \partial_s^\alpha \partial_\omega^\beta H = O(s^{-\alpha-1/2m})$$

which means, added to (T1), that $1 - H \in S_{1,0}^{-1/2m}$.

(It is straightforward to check that all the estimates given are uniform in the ω-variables.) Q.E.D.

We are going to use this foliation of $\mathbf{R}^n \backslash \{P(\xi) \leq a\}$ to estimate integrals involved in $\mathcal{F}(e^{itP(\xi)})$.

B. *Estimating integrals on* S^{n-1}. Let $\beta(s) \in C^\infty(\mathbf{R})$, $\beta \equiv 1$ for $s > a' + 1$ $\beta \equiv 0$ for $s < a'$, where $a' > a$ (the constant a is defined in Lemma 1). Let

$$x = ru, \quad r > 0, \ u \in S^{n-1}, \qquad \lambda = rs^{1/2m},$$

$$\phi(s,\omega) = s^{-1/2m}\rho(s,\omega)\langle u, \omega\rangle, \quad a_\beta(s,\omega) = \beta(s)\rho^{n-1}\frac{\partial \rho}{\partial s},$$

$$J_\beta(s,\lambda) = \int_{S^{n-1}} e^{i\lambda\phi(s,\omega)} a_\beta(s,\omega)\, d\omega.$$

The following lemma analyses the phase function ϕ:

LEMMA 3. *There exists a finite number of open sets $\Omega_i \subset S^{n-1}$ $(i = 1, \ldots, N)$ and a constant $d > a$ such that, for $s > d$:*

(1) On the complementary set of $\bigcup_i \Omega_i$ in S^{n-1}, $\phi(s, \omega)$ has no critical points in the ω-variables and $\|d_\omega \phi\| \geq C > 0$.

(2) On each Ω_i, $\phi(s, \omega)$ has only one critical point $\omega^i(s)$, $\omega^i(s) \in \Omega_i' \Subset \Omega_i$. At that point, $\phi(s, \omega)$ is nondegenerate: the eigenvalues of the Hessian matrix of ϕ in the ω-variables at $(s, \omega^i(s))$ have their modulus bounded from below, i.e. let $\tilde{H}(s) = \mathrm{Hess}_\omega(\phi)(s, \omega^i(s)) = d^2_{\omega\omega}\phi(s, \omega^i(s))$, then $\|\tilde{H}^{-1}(s)\| \leq C'$.

(3) The estimates are uniform in s, i.e. Ω_i, Ω_i', C and C' do not depend on $s > d$. $\omega^i \in C^\infty(]d, \infty[; S^{n-1})$.

PROOF.

$$\begin{aligned}\phi(s, \omega) &= p^{-1/2m}(\omega)\langle u, \omega\rangle + s^{-1/2m}\sigma(s, \omega)\langle u, \omega\rangle \\ &= p^{-1/2m}(\omega)\langle u, \omega\rangle + O(s^{-1/2m})\end{aligned}$$

uniformly on S^{n-1}, and this remains true under ω-differentiation, as shown in Lemma 2.

Hypothesis (H2) asserts that $\phi(\infty, \omega)$ has only nondegenerate critical points. Compactness of S^{n-1} and the fact that such points are isolated imply that this set of points is finite: $(\omega^i(\infty);\ i = 1, \ldots, N)$. The assumption (H2) asserts the nondegeneracy of $\phi(\infty, \omega)$ at these points: $d^2_{\omega\omega}\phi(\infty, \omega^i(\infty))$ invertible. This implies, by the implicit function theorem, the existence of $d' > 0$, O_i a neighborhood of $\omega^i(\infty)$ in S^{n-1} and $\omega^i(s) \in C^\infty(]d', \infty[; O_i)$ with

$$(d_\omega \phi(s, \omega) = 0 \Leftrightarrow \omega = \omega^i(s)) \quad \text{for } s > d' \text{ and } \omega \in O_i.$$

Moreover, invertibility of $d^2_{\omega\omega}\phi(\infty, \omega^i(\infty))$ implies the existence of open sets $\Omega_i \subset O_i$, where $d^2_{\omega\omega}\phi(\infty, \omega)$ is invertible and we have

$$\omega \in \Omega_i \Rightarrow \|(d^2_{\omega\omega}\phi(\infty, \omega))^{-1}\| < C'.$$

Lemma 2 asserts that

$$d^2_{\omega\omega}\phi(s, \omega) = d^2_{\omega\omega}\phi(\infty, \omega) + O(s^{-1/2m})$$

uniformly for $\omega \in S^{n-1}$. This implies the existence of $d'' > 0$ such that

$$(s > d'' \text{ and } \omega \in \Omega_i) \Rightarrow \|(d^2_{\omega\omega}\phi(s, \omega))^{-1}\| < 2C'.$$

This is uniform nondegeneracy of the phase function ϕ on $(\bigcup_i \Omega_i) \times]d'', \infty[$. Putting $\omega = \omega^i(s)$ gives

$$\|\tilde{H}^{-1}(s)\| < 2C' \quad \text{for } s > d''.$$

Let $\Omega_i' \Subset \Omega_i$ with $\omega^i(\infty) \in \Omega_i'$. On the complementary set of $\bigcup_i \Omega_i'$ in S^{n-1}, $d_\omega \phi(\infty, \omega) \neq 0$. Therefore there exists $d''' > 0$ such that

$$\left(s > d''' \text{ and } \omega \in \bigcap_i \complement \Omega_i'\right) \Rightarrow \|d_\omega \phi(s, \omega)\| > C.$$

We take $d = \max(d', d'', d''', a)$, where a is the constant defined in Lemma 1. Q.E.D.

We can therefore apply the Stationary Phase Theorem with parameters (Duistermatt [**5**]).

LEMMA 4. *For $s > d$, there exist absolute constants D_l and D_l' such that*
(a) *for $r > 0$*

$$\left|\frac{\partial^l}{\partial s^l}(e^{i\lambda\phi}a_\beta)\right| < D_l r^l s^{-1+(n/2m)-(l(2m-1)/2m)},$$

(b) *for $r > 1$*

$$\left|\frac{\partial^l}{\partial s^l}\int_{S^{n-1}} e^{i\lambda\phi}a_\beta\, d\omega\right| < D_l' \frac{s^{-1+((n+1)/4m)-(l(2m-1)/2m)}}{r^{-l+(n-1)/2}}.$$

PROOF. The first estimate is straightforward using the Leibniz rule to differentiate $(e^{i\lambda\phi}a_\beta)$, then Lemma 2 gives $\phi \in S^0_{1,0}$, $a_\beta \in S^{-1+n/2m}_{1,0}$ and $\lambda = rs^{1/2m}$ which implies inequality a.

To prove the second estimate, let $\alpha_i(\omega)$ $(i = 0, \dots, N)$ be a C^∞-partition of unity on S^{n-1} fitting the following covering: $(\Omega_0 = \bigcap_i \complement\overline{\Omega_i'}, \Omega_1, \dots, \Omega_N)$. The integral on S^{n-1} is then a sum of $N+1$ parts.

The integral over Ω_0 is rapidly decreasing when $\lambda \to \infty$, the phase ϕ being uniformly nonstationary (Lemma 3): As shown previously, we prove

$$\text{(R)} \qquad \frac{\partial^l}{\partial s^l}(e^{i\lambda\phi}a_\beta) = r^l e^{i\lambda\phi}a_{\beta,l}$$

with $a_{\beta,l} \in S^{-1+(n/2m)-(l(2m-1)/2m)}_{1,0}$. Then we use the fact that ϕ is nonstationary on Ω_0 to have a first order differential operator L in the ω-variables, with coefficients in $S^0_{1,0}(S^{n-1} \times \mathbf{R}_+)$ satisfying ${}^tL\phi = 1$. We use L k-times to integrate by parts in the ω-variables and (R) shows for $\lambda > h_1$ and any $k \in \mathbf{N}$:

$$\left|\frac{\partial^l}{\partial s^l}\int_{\Omega_0} e^{i\lambda\phi}a_\beta\, d\omega\right| < D_{0,k} r^l \lambda^{-k} s^{-1+(n/2m)-(l(2m-1)/2m)}.$$

To estimate integrals over Ω_i $(1 \le i \le N)$ we use the fact that, on Ω_i, the phase is stationary at a single point and nondegenerate at this point uniformly in s (Lemma 3). We apply the Stationary Phase Theorem with Parameters (Duistermaat [**5**]) and (R) to prove for $\lambda > h_2$:

$$\left|\frac{\partial^l}{\partial s^l}\int_{\Omega_i} e^{i\lambda\phi}a_\beta\, d\omega\right| < D_i r^{l-(n-1)/2} s^{-1+(n+1)/4m-l(2m-1)/2m}.$$

Taking $k \ge (n-1)/2$, $D = \max(D_i, D_{0,k})$ proves the estimate. Q.E.D.

C. *Proof of Theorem* 1. Let β be the previously defined function with $a' = d$. Let $\alpha(\xi) = 1 - \beta(P(\xi))$. We have

$$\begin{aligned}\mathcal{F}(e^{itP(\xi)}) &= \int_{\mathbf{R}^n} e^{i\langle x,\xi\rangle} e^{itP(\xi)}\alpha(\xi)\, d\xi + \int_0^\infty e^{its}\left(\int_{S^{n-1}} e^{i\lambda\phi}a_\beta\, d\omega\right) ds \\ &= I_1(t,x) + I_2(t,x)\end{aligned}$$

(a) $I_1(t,x)$ is the Fourier transform of a function in $\mathcal{D}(\mathbf{R}^n)$. It is rapidly decreasing in the x-variables and we have

$$\forall k \in \mathbf{N} \quad |I_1(t,x)| < K_k |t|^k r^{-k},$$

where K_k are absolute constants.

Taking $k = 0$ shows that $I_1(t,x) \in L^\infty_{\text{loc}}$ uniformly in t.

Taking k an integer, $k > n/q$, shows that $I_1(t,x) \in L^q(\mathbf{R}^n)$.
Let $c' > n/q$, $c' \in \mathbf{N}$. Then

$$\|I_1(t,\cdot)\|_{L^q(\mathbf{R}^n)} < K(1 + |t|^{c'}). \tag{J1}$$

(b) In order to estimate $I_2(t,x)$ locally, we integrate by parts in the s-variable l-times, and use Lemma 4(a) with $l > n/2m - 1$ to prove

(J2) $I_2(t,x) \in L^\infty_{\text{loc}}$ with bound less than $K'|t|^{-c}$ where c is an integer, $c > n/2m - 1$.

(c) In order to estimate $I_2(t,x)$ for large x, we first integrate by parts l'-times in the s-variable, as previously. Then apply Lemma 4(b) for $l' > (n+1)/2(2m-1)$ to obtain

$$|I_2(t,x)| < K'_{l'}|t|^{-l'} r^{l'-(n-1)/2}. \tag{J3}$$

Assumptions (H3) and $m > 2$ prove that there exists $l' \in \mathbf{N}$ with

$$(n+1)/2(2m-1) < l' \leq (n-1)/2.$$

Taking this value for l' shows that $I_2(t,\cdot) \in L^\infty(\mathbf{R}^n)$ with

$$\|I_2(t,\cdot)\|_{L^\infty} < K'_{l'}|t|^{-l'} + K'|t|^{-c}. \tag{J4}$$

Moreover, (J3) shows that $I_2(t,x) \in L^q(\mathbf{R}^n)$ for

$$(n-1)/2 - n/q > l'' > (n+1)/2(2m-1).$$

Assumption (H3′) implies that such an integer l'' exists for $q > q(m,n)$, and we have

$$\|I_2(t,\cdot)\|_{L^q} < K'|t|^{-c} + K''|t|^{-l''}. \tag{J5}$$

(d) We finally notice that we can choose l' and l'' less or equal to c to rewrite (J1), (J2), (J4) and (J5):

$$\|\mathcal{F}(e^{itP(\xi)})\|_{L^\infty} < C_\infty(1 + |t|^{-c}),$$
$$\|\mathcal{F}(e^{itP(\xi)})\|_{L^q} < C_q(|t|^{c'} + |t|^{-c}). \quad \text{Q.E.D.}$$

REMARK. This computation shows that $I_2(t,x)$ is rapidly decreasing as $|t| \to \infty$. But in the general case, $I_1(t,x)$ does not decay as $|t| \to \infty$. Under an additional assumption, this will be the case locally.

COROLLARY. ***Local decay.*** *If* (H1), (H2) *and* (H3) *are fulfilled and if we have*

(H4) *for* $\|\xi\| < a$, $P(\xi)$ *is nondegenerate at its critical points,*
then, if B *is a bounded set in* $\mathbf{R}^n$, *we have for large* $|t|$

$$\forall x \in B \quad |\mathcal{F}(e^{itP(\xi)})(x)| < C|t|^{-n/2}.$$

PROOF. Using Lemma 4(a), we prove the rapid decay of $I_2(t,x)$ for bounded x. To estimate $I_1(t,x)$, we consider it as an oscillatory integral on a compact set with phase $P(\xi)$ and the parameter $|t| \to \infty$. (H4) enables us to apply the Stationary Phase Theorem which gives an estimate by $|t|^{-n/2}$. Q.E.D.

3. $(L^p, L^{p'})$ **and** (L^p, L^q) **estimates** $(1 \leq p \leq 2)$. Let $(e^{itP(D)}U_0)(x)$ be the solution $U(t,x)$ of the Cauchy problem $(**)$. We have the following $(L^p, L^{p'})$ and (L^p, L^q) estimates for $e^{itP(D)}$.

THEOREM 2. *Assume P satisfies* (H1), (H2) *and* (H3). *Then for any $t \neq 0$ and any p, $1 \leq p \leq 2$, $e^{itP(D)}$ maps continuously $L^p(\mathbf{R}^n)$ into $L^{p'}(\mathbf{R}^n)$ and we have the estimate*

$$\|e^{itP(D)}\|_{\mathcal{L}(L^p, L^{p'})} < C(1 + |t|^{-c\theta}),$$

where $\theta = p^{-1} - p'^{-1}$, $p^{-1} + p'^{-1} = 1$, c is an integer, $c > n/2m - 1$, and C does not depend on t.

THEOREM 2′. *Assume P satisfies* (H1), (H2) *and* (H3′). *If $1 < p \leq 2$ and $q(m,n,p) < q \leq p'$, then $e^{itP(D)}$ continuously maps $L^p(\mathbf{R}^n)$ into $L^q(\mathbf{R}^n)$ for $t \neq 0$, and we have the estimate*

$$\|e^{itP(D)}\|_{\mathcal{L}(L^p, L^q)} < C'(|t|^{c'\theta} + |t|^{-c\theta}).$$

Here
$\theta = p^{-1} - p'^{-1}$,
c is an integer with $c > n/2m - 1$,
c' is an integer with $c' > n(p' - q)/q(p' - 2)$, and
$(q(m,n,p))^{-1} = p'^{-1} + \theta(q(m,n))^{-1}$, where $q(m,n)$ is given in Theorem 1.

PROOF. Since $e^{itP(\xi)}$ is of modulus one, $e^{itP(D)}$ is continuous in $L^2(\mathbf{R}^n)$ with norm equal to one. Theorem 1 gives the continuity of $e^{itP(D)}$ from $L^1(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$, $q(m,n) < q \leq \infty$. Then the Riesz Thorin interpolation theorem (Stein [**11**], Lions and Peetre [**9**]) proves the estimate. Q.E.D.

REMARK 2. With p and q fulfilling the same assumptions, if $P(\xi)$ is homogeneous, the estimates can be improved to

$$\|e^{itP(D)}\|_{\mathcal{L}(L^p, L^q)} < C_{p,q} t^{-(n/2m)(1/p - 1/q)}.$$

We now come to $(L^p, L^{p'})$ and (L^p, L^q) estimates of the resolvent operator of $iP(D)$:

THEOREM 3. *Assume P satisfies* (H1), (H2) *and* (H3). *Then for $\lambda \in \mathbf{C}$ with* $\operatorname{Re}\lambda \neq 0$, *and for p, $2c/c+1 < p \leq 2$, we have*

$$\|(\lambda - iP(D))^{-1}\|_{\mathcal{L}(L^p, L^{p'})} < C|\operatorname{Re}\lambda|^{-1}(1 + |\operatorname{Re}\lambda|^{c\theta}),$$

where $\theta = p^{-1} - p'^{-1}$, c is an integer, $c > n/2m - 1$, and C does not depend on λ.

THEOREM 3′. *Assume P satisfies* (H1), (H2) *and* (H3′). *Then for $\lambda \in \mathbf{C}$,* $\operatorname{Re}\lambda \neq 0$, *for p, $2c/c+1 < p \leq 2$, and for q, $q(m,n,p) < q \leq p'$, we have*

$$\|(\lambda - iP(D))^{-1}\|_{\mathcal{L}(L^p, L^q)} < C|\operatorname{Re}\lambda|^{-1}(|\operatorname{Re}\lambda|^{-c'\theta} + |\operatorname{Re}\lambda|^{+c\theta}),$$

where the parameters involved have the same values as in Theorem 2′.

REMARK 3. If $2m > n + 1$, then $c = 1$ and these estimates are valid for any p, $1 < p \leq 2$.

PROOF. With the same notation as before, we consider for $\operatorname{Re}\lambda > 0$

$$F(\lambda, x) = \int_0^\infty e^{-\lambda t} U(t,x)\, dt.$$

For $U_0 \in L^p(\mathbf{R}^n)$ with $2 \geq p > 2c/c+1$, the integral is convergent in L^q for any q, $q(m,n,p) < q \leq p'$, in view of Theorems 2 and 2'. This leads to

$$\|F(\lambda, x)\|_{L^q(\mathbf{R}^n)} \leq C|\mathrm{Re}\,\lambda|^{-1}(|\mathrm{Re}\,\lambda|^{-c'\theta} + |\mathrm{Re}\,\lambda|^{c\theta})\|U_0\|_{L^p(\mathbf{R}^n)}.$$

Then we take $U_0 \in S(\mathbf{R}^n)$, we compute $iP(D)F(\lambda, x)$ by passing the operator under the integral sign, and then integrate by parts to prove

$$F(\lambda, x) = (\lambda - iP(D))^{-1}U_0.$$

A density argument ends the proof for $\mathrm{Re}\,\lambda > 0$.

For $\mathrm{Re}\,\lambda < 0$, the same proof is valid with

$$F(\lambda, x) = -\int_{-\infty}^{0} e^{\lambda t}U(t, x)\,dt. \quad \text{Q.E.D.}$$

REMARK 4. For homogeneous P we can take $1 < p \leq 2$, and the bound can be improved to

$$\|(\lambda - iP(D))^{-1}\|_{\mathcal{L}(L^p, L^q)} < C|\mathrm{Re}\,\lambda|^{-1+(n/2m)(1/p-1/q)}.$$

4. (L^p, L^p) estimates and smooth distribution groups $(1 < p < \infty)$.

A. $e^{itP(D)}$ *as a distribution with values in* $\mathcal{L}(L^p)$. In order to prove (L^p, L^p) estimates for $e^{itP(D)}$, we first recall that $e^{itP(D)}$ is not a continuous mapping from $L^p(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ unless $p = 2$, or $t = 0$, or $P(D)$ is a first order differential operator (Hörmander [**7**] and Brenner [**4**]). We will prove that for any $\phi \in \mathcal{D}(\mathbf{R})$ (the Schwartz space), the following operator is continuous in $L^p(\mathbf{R}^n)$:

$$\mathcal{G}(\phi) = \int_{-\infty}^{+\infty} \phi(t)e^{itP(D)}\,dt.$$

(We will denote it by $\mathcal{G}_p(\phi)$ when it will operate in $L^p(\mathbf{R}^n)$.) So we consider $e^{itP(D)}$ as a distribution in the t-variable with values in $\mathcal{L}(L^p(\mathbf{R}^n))$, and estimate its order.

The only hypothesis we will assume for $P(D)$ throughout §4 will be

(HP) $P(D)$ is a real valued, elliptic polynomial of order $2m$, $P(\xi) \neq 0$ for $\xi \neq 0$.

REMARK. This last assumption about the zeros of $P(\xi)$ is unnecessary: one can always add to any real elliptic polynomial a constant c such that $P(\xi) + c$ fulfills (HP). Adding this constant changes in an obvious way the subsequent estimates. We adopt it in order to simplify notations.

DEFINITION 1. For $l \in \mathbf{N}$, let p_l be the following norm on $\mathcal{D}(\mathbf{R})$:

$$\forall \phi \in \mathcal{D}(\mathbf{R}) \quad p_l(\phi) = \sum_{0<k<l} \left\| t^k \frac{d^k\phi}{dt^k} \right\|_{L^1(\mathbf{R})}.$$

Let T_k denote the completion of $\mathcal{D}(\mathbf{R})$ for p_k, and T_∞ the completion of $\mathcal{D}(\mathbf{R})$ for the family $(p_l)_{l\in\mathbf{N}}$. We will denote by T_k^+ and p_k^+ the same objects with $\mathcal{D}(\mathbf{R})$ replaced by $\mathcal{D}(\mathbf{R}_+)$.

THEOREM 4. *Let $P(D)$ satisfy* (HP). *Then for any p, $1 < p < \infty$, and any $k \in \mathbf{N}$, $k > n/2$, $\mathcal{G}_P$ is a continuous linear mapping from T_k to $\mathcal{L}(L^p(\mathbf{R}^n))$.*

PROOF. For any $U_0 \in S(\mathbf{R}^n)$, the Cauchy problem (**) has a unique solution $U(t,x) = (e^{itP(D)}U_0)(x)$ which belongs to $S(\mathbf{R}^n)$ for any fixed t. Let $\phi(t) \in \mathcal{D}(\mathbf{R})$. The following computation is obvious in $S(\mathbf{R}^n)$:

$$\mathcal{G}(\phi)U_0 = \int_{-\infty}^{\infty} \phi(t)e^{itP(D)}U_0\,dt = \int_{-\infty}^{\infty} \phi(t)\mathcal{F}(e^{itP(\xi)}\mathcal{F}U_0(\xi))\,dt$$
$$= \mathcal{F}\left(\int_{-\infty}^{\infty} \phi(t)e^{itP(\xi)}\,dt \cdot \mathcal{F}U_0(\xi)\right) = \mathcal{F}(\hat{\phi}(P(\xi)) \cdot \mathcal{F}U_0(\xi)),$$

where $\hat{\phi}$ denotes the inverse Fourier transform of ϕ in the variable t. To prove Proposition 1, we have to prove that $\hat{\phi}(P(\xi)) \in M_p$, the space of Fourier multipliers in $L^p(\mathbf{R}^n)$, and the M_p norm of $\hat{\phi}(P(\xi))$ is bounded by the T_k norm of ϕ. A density argument will conclude the proof. We will use a sufficient condition for a function to belong to M_p given by Stein [**11**]: for every differential monomial ∂_ξ^α with $|\alpha| \le k$ $(k > n/2)$, $|\partial_\xi^\alpha \hat{\phi}(P(\xi))|$ must be bounded by $p_{|\alpha|}(\phi)\|\xi\|^{-|\alpha|}$. This is done inductively. For $|\alpha| = 0$, we have

$$\forall \phi \in \mathcal{D}(\mathbf{R}),\ \forall \xi \in \mathbf{R}^n, \quad |\hat{\phi}(P(\xi))| \le \|\hat{\phi}\|_{L^\infty} \le \|\phi\|_{L^1} = p_0(\phi).$$

Assume that for any $\beta \in \mathbf{N}^n$ with $|\beta| < |\alpha|$, we have the following estimate for the derivative of order β:

$$\forall \phi \in \mathcal{D}(\mathbf{R}),\ \forall \xi \in \mathbf{R}^n, \quad |\partial_\xi^\beta \hat{\phi}(P(\xi))| \le C p_{|\beta|}(\phi)\|\xi\|^{-|\beta|}.$$

Then, with a little abuse of notation, and using the classical formulas of derivatives of Fourier transforms and Fourier transforms of derivatives (taken at the point $P(\xi)$), we compute the derivative of order α:

$$\begin{aligned}\partial_\xi^\alpha \hat{\phi}(P(\xi)) &= \partial_\xi^{\alpha-1}[i(\partial_\xi P) \cdot \widehat{t\phi}(P(\xi))] \\ &= \partial_\xi^{\alpha-1}[(P^{-1} \cdot \partial_\xi P) \cdot (\hat{\phi} + \widehat{t\phi'})(P(\xi))] \\ &= \sum C_{\alpha-1}^j \partial_\xi^j (P^{-1} \cdot \partial_\xi P) \cdot \partial_\xi^{\alpha-1-j}(\hat{\phi} + \widehat{t\phi'})(P(\xi))\end{aligned}$$

by the Leibniz rule. Using the assumed estimates we have

$$\begin{aligned}|\partial_\xi^\alpha \hat{\phi}(P(\xi))| &\le \sum C_{\alpha-1}^j |\partial_\xi^j (P^{-1} \cdot \partial_\xi P)| \cdot p_{|\alpha-1-j|}(\phi + t\phi') \cdot \|\xi\|^{-|\alpha-1-j|} \\ &\le \|\xi\|^{-|\alpha|} \sum C_{\alpha-1}^j \cdot \sup_{\xi \in \mathbf{R}^n} (\|\xi\|^{|j|+1} \cdot |\partial_\xi^j (P^{-1} \cdot \partial_\xi P)|) \cdot p_{|\alpha-1-j|}(\phi + t\phi').\end{aligned}$$

Boundedness of $\|\xi\|^{|j|+1}\partial_\xi^j (P^{-1} \cdot \partial_\xi P)$ follows from (HP). Using the inequality $p_{|\gamma-1|}(t\phi') < D_\gamma p_{|\gamma|}(\phi) < D_\gamma p_{|\alpha|}(\phi)$ for $|\alpha| \ge |\gamma|$ we have

$$|\partial_\xi^\alpha \hat{\phi}(P(\xi))| \le C' p_{|\alpha|}(\phi)\|\xi\|^{-|\alpha|}. \quad \text{Q.E.D.}$$

REMARK 5. Following [**2**], if P is homogeneous, $\mathcal{G}_p$ is a continuous mapping from T_k to $\mathcal{L}(L^p(\mathbf{R}^n))$ if $k > n|1/p - 1/2|$.

COROLLARY. *Assume $P(D)$ satisfies* (HP). *Let $iP_p(D)$ be the densely defined, closed operator in $L^p(\mathbf{R}^n)$ defined as $iP(D)$ with domain $W^{2m,p}(\mathbf{R}^n)$. For any $\lambda \in \mathbf{C}$, $\operatorname{Re}\lambda \neq 0$, $(\lambda - iP_p(D))$ has a bounded inverse in $L^p(\mathbf{R}^n)$, $1 < p < \infty$, and we have the estimate*

$$\|(\lambda - iP_p(D))^{-1}\|_{\mathcal{L}(L^p(\mathbf{R}^n))} < C|\operatorname{Re}\lambda|^{-1}\,|\lambda(\operatorname{Re}\lambda)^{-1}|^{(n+3)|1/p-1/2|}.$$

PROOF. We first notice that for any $k \in \mathbf{N}$ and $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda > 0$, $Y(t)e^{-\lambda t}$ belongs to T_k with its p_k norm bounded by $C_k|\lambda|^k\,|\operatorname{Re}\lambda|^{-k-1}$. So, for any positive ε, $\mathcal{G}(Y(t)e^{-\lambda t})$ extends to a bounded operator in $L^{1+\varepsilon}(\mathbf{R}^n)$ with norm less than $C_k|\lambda|^k|\operatorname{Re}\lambda|^{-k-1}$, $k > n/2$.

On the other hand, $\mathcal{G}(Y(t)e^{-\lambda t})$ is obviously bounded in $L^2(\mathbf{R}^n)$ with norm less than $|\operatorname{Re}\lambda|^{-1}$. Interpolation between $L^{1+\varepsilon}$ and L^2 gives $\mathcal{G}(Y(t)e^{-\lambda t})$ bounded in $L^p(\mathbf{R}^n)$, $1+\varepsilon \leq p \leq 2$, with norm less than

$$|\operatorname{Re}\lambda|^{-1}|\lambda(\operatorname{Re}\lambda)^{-1}|^{(E(n/2)+1)(1+\varepsilon)(p^{-1}-p'^{-1})/(1-\varepsilon)}$$

which is always less than $|\operatorname{Re}\lambda|^{-1}|\lambda(\operatorname{Re}\lambda)^{-1}|^{(n+3)(p^{-1}-2^{-1})}$. An adjointness argument gives the same result for $1+\varepsilon \leq p' \leq 2$. On $S(\mathbf{R}^n)$, we have

$$(\lambda - iP(D))\mathcal{G}(Y(t)e^{-\lambda t}) = \mathcal{G}(Y(t)e^{-\lambda t})(\lambda - iP(D)) = I.$$

This proves that $\mathcal{G}_p(Y(t)e^{-\lambda t})$ is the inverse operator of the operator $(\lambda - iP(D))$ in $L^p(\mathbf{R}^n)$ with domain

$$\{U_0 \in L^p(\mathbf{R}^n) \text{ with } P(D)U_0 \in L^p(\mathbf{R}^n)\}.$$

But this is exactly $W^{2m,p}(\mathbf{R}^n)$, following the fact that

$$\partial_\xi^\alpha U_0 = \overline{\mathcal{F}}(\xi^\alpha \mathcal{F} U_0) = \overline{\mathcal{F}}(\xi^\alpha P(\xi)^{-1}\mathcal{F}(P(D)U_0))$$

and using hypothesis (HP) and the multipliers theorem of Stein [**11**] quoted above to prove that $\xi^\alpha P(\xi)^{-1}$ is a multiplier of $L^p(\mathbf{R}^n)$ if $|\alpha| \leq 2m$. For $\operatorname{Re}\lambda < 0$, the same proof is valid with $\mathcal{G}(-Y(-t)e^{\lambda t})$ in place of $\mathcal{G}(Y(t)e^{-\lambda t})$.

REMARK. For p close to 1 or to infinity, and small values of n, a better estimate could be proved directly using Theorem 4 in $L^p(\mathbf{R}^n)$ to get

$$\|(\lambda - iP(D))^{P-1}\|_{\mathcal{L}(L^p)} < C|\operatorname{Re}\lambda|^{-1}|\lambda(\operatorname{Re}\lambda)^{-1}|^k \quad \text{with } k > n/2.$$

In order to study the case $iP(D) + V(x)$, we will need an inverse result. We now introduce the abstract framework which will allow us to prove it.

C. *Smooth distribution semigroups on a Banach space.* Smooth distribution groups are a special case of distribution semigroups introduced by Lions [**8**]. They turn out to be the right tool to analyse differential operators in $L^p(\mathbf{R}^n)$. A particular class of these distributions was studied by us in [**2**]. We will not give those proofs here which are slight modifications of proofs given in [**2**].

DEFINITION 2. Let X be a Banach space. A smooth distribution semigroup of order $k \in \mathbf{N}$ and exponential growth $\delta > 0$ is a linear mapping $\mathcal{G}_+$ from $\mathcal{D}(\mathbf{R}_+)$ to $\mathcal{L}(X)$ such that

(0) $e^{-\delta t}\mathcal{G}_+$ extends continuously to T_k^+: $\forall\phi \in \mathcal{D}(\mathbf{R}_+)$, $\|\mathcal{G}_+(e^{-\delta t}\phi)\|_{\mathcal{L}(X)} < Cp_k(\phi)$.

(i) $\forall\phi \in T_k^+$, $\forall\psi \in T_k^+$, $\mathcal{G}_+(\phi * \psi) = \mathcal{G}_+(\phi)\mathcal{G}_+(\psi)$.

(ii) There exists an everywhere dense subspace D of X such that for every $x \in D$ the distribution $\mathcal{G}_+ \otimes x$ is a continuous function on $\overline{\mathbf{R}}_+$, with value x at the origin.

NOTATION. The class of smooth distribution semigroups of order k and exponential growth δ will be denoted by $\sigma_+(k,\delta)$.

REMARK 6. (a) (i) makes sense because T_k^+ is an algebra for the (additive) convolution.

(b) We notice that for any $\phi \in \mathcal{D}(\mathbf{R}_+)$, $(s^{-1}\phi(s^{-1}\cdot))_{s>0}$ is a bounded subset of T_k^+. The Ascoli theorem then shows that $e^{-\delta t}\mathcal{G}_+(s^{-1}\phi(s^{-1}\cdot))$ converges strongly to the identity. This implies

$$N = \bigcap_{\phi\in T^+_{k,\delta}} \operatorname{Ker} \mathcal{G}_+(\phi) = \{0\},$$

$$R = \bigcup_{\phi\in T^+_{k,\delta}} \operatorname{Im} \mathcal{G}_+(\phi) \quad \text{is everywhere dense in } X.$$

Let $\mathcal{G}_+(-\delta')$ be the operator defined on R by

$$\mathcal{G}_+(-\delta')\mathcal{G}_+(\phi)x = \mathcal{G}_+(-\phi')x \quad \text{for } x \in X,\ \phi \in T_k^+.$$

The properties quoted in Remark 6 show that this definition is consistent and that $\mathcal{G}_+(-\delta')$ is closable.

DEFINITION 3. The infinitesimal generator of $\mathcal{G}_+ \in \sigma_+(k,\delta)$ is the closure of $\mathcal{G}_+(-\delta')$.

Spectral properties of generators of smooth distribution semigroups are summarized in the following:

PROPOSITION 1. *If A generates $\mathcal{G}_+ \in \sigma_+(k,\delta)$ on a Banach space X, then A is densely defined, closed, and for any $\lambda \in \mathbf{C}$, $\operatorname{Re}\lambda > \delta$, λ belongs to the resolvent set of A. For any $\mu \in \mathbf{C}$ with $\operatorname{Re}\mu > 0$, we have*

$$\|(\lambda I - A)^{-\mu}\|_{\mathcal{L}(X)} < C(\mu)|\lambda-\delta|^k(\operatorname{Re}\lambda - \delta)^{-k-\operatorname{Re}\mu},$$

where $C(\mu) = C\Gamma(\operatorname{Re}\mu)|\Gamma(\mu)|^{-1}\,|\mu|^k$.

PROOF. We will sketch it for μ a positive integer. The proof for $\mu \in \mathbf{C}$ is a slight modification of that given in [**2**]. We first notice that for $\operatorname{Re}\lambda > \delta$, $Y(t)t^{\mu-1}e^{-(\lambda-\delta)t}$ belongs to T_k^+ and it is easy to see that

$$T(\mu)(\lambda I - A)^{-\mu} = \mathcal{G}_+(Y(t)t^{\mu-1}e^{-\lambda t}).$$

Then, $e^{-\delta t}\mathcal{G}_+$ being bounded on T_k^+ gives the estimate if we compute

$$p_k(Y(t)t^{\mu-1}e^{-(\lambda-\delta)t}) < C|\mu|^k|\lambda-\delta|^k(\operatorname{Re}\lambda-\delta)^{-k-\mu}. \quad \text{Q.E.D.}$$

The inverse result is the following Hille-Yosida type estimate.

PROPOSITION 2. *Let A be a closed, densely defined operator in a Banach space X. If for any $\lambda \in \mathbf{C}$, $\operatorname{Re}\lambda > \delta$, λ belongs to the resolvent set of A and we have the estimate*

$$\|(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} < C|\lambda-\delta|^k(\operatorname{Re}\lambda-\delta)^{-k-1},$$

then A generates a smooth distribution semigroup $\mathcal{G}_+$ of order $k+2$ and exponential growth δ.

PROOF. For any $\phi \in \mathcal{D}(\mathbf{R}_+)$ let $\tilde{\phi}$ denote its Laplace transform

$$\tilde{\phi}(\lambda) = \int_0^\infty e^{\lambda t}\phi(t)\,dt.$$

Let $\Gamma(c)$ be the line $\operatorname{Re}\lambda = c$ positively oriented in the direction of increasing $\operatorname{Im}\lambda$. We define $\mathcal{G}_+$ by

$$\mathcal{G}_+(\phi) = \frac{1}{2i\pi}\int_{\Gamma(\delta+\varepsilon)} \tilde{\phi}(\lambda)(\lambda - A)^{-1}\,d\lambda.$$

This integral is obviously convergent in $\mathcal{L}(X)$. Standard holomorphic calculus shows point (i) of Definition 2. The resolvent identity to order $k+2$ shows that $D(A^{k+2})$ can be taken for the dense subspace of point (ii) of Definition 2. It is straightforward to check that A is the infinitesimal generator of $\mathcal{G}_+$.

It remains to show that $e^{-\delta t}\mathcal{G}_+$ extends continuously to T_k^+. Using Fubini's Theorem, one has

$$\begin{aligned}
\mathcal{G}_+(e^{-\delta t}\phi) &= \frac{1}{2i\pi}\int_{\Gamma(\varepsilon)} \tilde{\phi}(\lambda)(\lambda + \delta - A)^{-1}\,d\lambda \\
&= \int_0^\infty t^{k+2}\phi^{(k+2)}(t)\left(\frac{1}{2i\pi}\int_{\Gamma(\varepsilon)} (\lambda t)^{-k-2}e^{\lambda t}(\lambda + \delta - A)^{-1}\,d\lambda\right)dt \\
&= \int_0^\infty t^{k+2}\phi^{(k+2)}(t)\left(\frac{1}{2i\pi}\int_{\Gamma(t\varepsilon)} \lambda^{-k-2}e^{\lambda}(t^{-1}\lambda + \delta - A)^{-1}t^{-1}\,d\lambda\right)dt,
\end{aligned}$$

and we only need to show that the norm of the path integral in $\mathcal{L}(X)$ is finite and does not depend on t. We change the integration path to $\Gamma(1)$ and use the estimate assumed on the resolvent operator to end the proof. Q.E.D.

We have the following regularity result for $\mathcal{G}_+ \in \sigma_+(k,\delta)$ on $D(A^k)$. Its proof is similar to that given in [**2**].

PROPOSITION 3. *Let $\mathcal{G}_+ \in \sigma_+(k,\delta)$ on a Banach space X. Let A be its infinitesimal generator. For any $x \in D(A^k)$, the distribution $\mathcal{G}_+ \otimes x$ is a function on $\overline{\mathbf{R}}_+$, denoted by $e^{tA}x$, and we have the estimate*

$$\forall t \geq 0, \quad \|e^{tA}x\| < (\|x\| + \|A^k x\|)(1 + t^k)e^{\delta t}.$$

DEFINITION 4. Let X be a Banach space and A a linear operator in X. A generates a smooth distribution group $\mathcal{G}$ of order k and exponential growth δ (in short $\mathcal{G} \in \sigma(k,\delta)$) if A and $-A$ generate elements of $\sigma_+(k,\delta)$.

D. *Constant coefficients evolution equations in $L^p(\mathbf{R}^n)$* $(1 < p < \infty)$.

THEOREM 5. *Let $P(D)$ be a differential operator satisfying* (HP) *and $\mathcal{G}_p$ the distribution associated to $e^{itP(D)}$ in $L^p(\mathbf{R}^n)$. $\mathcal{G}_p$ is a smooth distribution group of order $k > n/2$ and exponential growth* 0 *in $L^p(\mathbf{R}^n)$. Its infinitesimal generator is $iP(D)$ with domain $W^{2m,p}(\mathbf{R}^n)$.*

PROOF. This is a rewriting of Theorem 4. The only thing to be computed is the infinitesimal generator A_p of $\mathcal{G}_p$. Following the proof of the corollary of Theorem 4, it is enough to show that the domain of A_p is $\{U_0 \in L^p(\mathbf{R}^n)$ with $iP(D)U_0 \in L^p(\mathbf{R}^n)\}$ and that, on R, we have $A_p = iP(D)$.

For $\phi \in T_k^+$ and $U_0 \in L^p(\mathbf{R}^n)$ we have

$$\begin{aligned}
A_p\mathcal{G}_p(\phi)U_0 &= \mathcal{G}_p(-\phi')U_0 = \overline{\mathcal{F}}(\widehat{-\phi'}(P(\xi))\mathcal{F}U_0) \\
&= \overline{\mathcal{F}}(iP(\xi)\hat{\phi}(P(\xi))\mathcal{F}U_0) = iP(D)\mathcal{G}_p(\phi)U_0.
\end{aligned}$$

If $U_0 \in L^p(\mathbf{R}^n)$ and $iP(D)U_0 \in L^p(\mathbf{R}^n)$, let $\phi \in \mathcal{D}(\mathbf{R}_+)$ with $\int \phi = 1$ and let $\phi_s = s^{-1}\phi(s^{-1}\cdot)$. Then $\mathcal{G}_p(\phi_s)U_0$ converges to U_0 in $L^p(\mathbf{R}^n)$ because $\widehat{\phi_s}(P(\xi))$ converges to one in M_p, applying Stein [**11**], and $\mathcal{G}_p(-\phi_s')U_0 = \mathcal{G}_p(\phi_s)(iP(D)U_0)$ converges in $L^p(\mathbf{R}^n)$ to $iP(D)U_0$. So $D(A_p) \supset W^{2m,p}(\mathbf{R}^n)$. For the inverse inclusion we just note that $\mathcal{G}_p(-\phi_s')U_0$ converges to $iP(D)U_0$ in $\mathcal{D}'(\mathbf{R}^n)$ if $U_0 \in D(A_p)$. So $iP(D)U_0 \in L^p(\mathbf{R}^n)$. Q.E.D.

REMARK 7. (a) Using the corollary of Theorem 4 and Proposition 2, we can improve the order of $\mathcal{G}_p$ in $L^p(\mathbf{R}^n)$ to $k > 2 + (n+3)|1/p - 1/2|$.

(b) Theorem 5 and Proposition 1 improve the estimate of the corollary of Theorem 4. They give an estimate for the powers of the resolvent of $iP_p(D)$.

(c) If P is homogeneous, the order of $\mathcal{G}_p$ can be improved to $k > n|1/p - 1/2|$.

PROPOSITION 4. *For $U_0 \in W^{2mk,p}(\mathbf{R}^n)$, $k > n/2$, the solution of the Cauchy problem (**) is a continuous function in the t-variable with values in $L^p(\mathbf{R}^n)$. We have the estimate*

$$\|e^{itP(D)}U_0\|_{L^p(\mathbf{R}^n)} < C(1 + |t|^k)\|U_0\|_{W^{2mk,p}(\mathbf{R}^n)}.$$

PROOF. This is a translation of Proposition 3 using Theorem 5.

5. The Cauchy problem $\partial_t - iP(D) - V(x)$: (L^p, L^p) estimates. We consider the Cauchy problem

$$(*) \qquad \partial U/\partial t = (iP(D) + V(x))U; \quad U(0,x) = U_0(x) \in L^p(\mathbf{R}^n).$$

We are now in position to prove that under the subsequent assumptions on P and V, the solution is a distribution in the t-variable with values in $L^p(\mathbf{R}^n)$. The order of this distribution is any integer k with $k > (n+3)|p^{-1} - 2^{-1}| + 4$. This will imply a precise estimate in $\mathcal{L}(L^p(\mathbf{R}^n))$ of the resolvent operator of $iP(D) + V(x)$.

Here $iP(D) + V(x)$ will mean this differential operator with domain $\{U_0 \in W^{2m,p}(\mathbf{R}^n)$ with $VU_0 \in L^p(\mathbf{R})\}$.

Let c be an integer with $c > n/2m - 1$.

Let $q(m,n) = n(2m-1)/((m-1)(n-3) - 2)$.

Let $q'(m,n)$ be the conjugate index $q(m,n)^{-1} + (q'(m,n))^{-1} = 1$.

Let $\tilde{q} = q'(m,n)$.

THEOREM 6. *Assume that* (i) $2c/(c+1) < p < 2c/(c-1)$.

(ii) $P(D)$ *satisfies* (H1), (H2), (H3$'$) *and* (HP).

(iii) $V = V_1 + V_2$ *with* $V_1 \in L^{r_1}(\mathbf{R}^n)$, $r_1^{-1} = |p^{-1} - p'^{-1}|$, $V_2 \in L^{r_2}(\mathbf{R}^n)$, $\tilde{q}^{-1}|p^{-1} - p'^{-1}| < r_2^{-1} \leq |p^{-1} - p'^{-1}|$.

Then for any integer k, $k > (n+3)|p^{-1} - 2^{-1}| + 2$, and some $\delta > 0$, $iP(D) + V(x)$ generates a smooth distribution group in $L^p(\mathbf{R}^n)$ of order k and exponential growth δ.

PROOF. (a) First we note that $iP(D) + V$ and $-(iP(D) + V)$ satisfy the same assumptions. So we just have to prove that $iP(D) + V$ generates a smooth distribution semigroup of order k and exponential growth δ.

(b) For $p \leq 2$, we write the resolvent operator, whenever it exists, in the form

$$(\lambda - (iP(D) + V))^{-1} = (I - V(\lambda - iP(D))^{-1})^{-1}(\lambda - iP(D))^{-1}.$$

By the Neumann series, $(I - V(\lambda - iP(D))^{-1})^{-1}$ will exist and be bounded if $\|V(\lambda - iP(D))^{-1}\|_{\mathcal{L}(L^p)} < 1$. But

$$\begin{aligned}\|V(\lambda - iP(D))^{-1}\|_{\mathcal{L}(L^p)} &\leq \|V_1\|_{\mathcal{L}(L^{p'}, L^p)}\|(\lambda - iP(D))^{-1}\|_{\mathcal{L}(L^p, L^{p'})} \\ &\quad + \|V_2\|_{\mathcal{L}(L^{s_2}, L^p)}\|(\lambda - iP(D))^{-1}\|_{\mathcal{L}(L^p, L^{s_2})} \\ &\leq C\|V_1\|_{L^{r_1}}|\operatorname{Re}\lambda|^{-1}(1 + |\operatorname{Re}\lambda|^{c\theta}) \\ &\quad + C\|V_2\|_{L^{r_2}}|\operatorname{Re}\lambda|^{-1}(|\operatorname{Re}\lambda|^{-c'\theta} + |\operatorname{Re}\lambda|^{c\theta})\end{aligned}$$

by Theorems 3 and 3′. Following these theorems, we must have

$2c/(c+1) < p \leq 2$ and $r_1^{-1} = p^{-1} - p'^{-1}$,

$q(m,n,p) < s_2 \leq p'$ and $r_2^{-1} = p^{-1} - s_2^{-1}$ so $p^{-1} - (q(m,n,p))^{-1} < r_2^{-1} \leq p^{-1} - p'^{-1}$,

$\theta = p^{-1} - p'^{-1}$,

c is an integer with $c > n/2m - 1$, and

c' is an integer with $c' > n(p' - s_2)/s_2(p' - 2)$.

Finally we have for $|\operatorname{Re}\lambda| > 1$

$$\|V(\lambda - iP(D))^{-1}\|_{\mathcal{L}(L^p)} < C(\|V_1\|_{L^{r_1}} + \|V_2\|_{L^{r_2}})(1 + |\operatorname{Re}\lambda|^{c\theta})|\operatorname{Re}\lambda|^{-1}.$$

This must be strictly less than one for large $|\operatorname{Re}\lambda|$: we must add the condition $c\theta < 1$, and this is true when $p > 2c/(c+1)$. Thus there exists some positive δ such that for $|\operatorname{Re}\lambda| > \delta$, $\|V(\lambda - iP(D))^{-1}\|_{\mathcal{L}(L^p)} < 1/2$. In this case, we have the following estimate for the resolvent by the corollary of Theorem 4:

$$\|(\lambda - (iP(D) + V))^{-1}\|_{\mathcal{L}(L^p(\mathbf{R}^n))} < 2C|\operatorname{Re}\lambda|^{-1}|\lambda(\operatorname{Re}\lambda)^{-1}|^{(n+3)|p^{-1} - 2^{-1}|}.$$

This implies, by the abstract Proposition 2, that $iP(D) + V$ generates a smooth distribution semigroup of order any integer k with $k > (n+3)(p^{-1} - 2^{-1}) + 2$ and of exponential growth δ. All encountered assumptions are the assumptions given in the statement of the theorem for $p \leq 2$, but the assumption on r_2. The condition $p^{-1} - (q(m,n,p))^{-1} < r_2^{-1}$ is equivalent to $(p^{-1} - p'^{-1})\tilde{q}^{-1} < r_2^{-1}$ by definition of $q(m,n,p)$

$$\begin{aligned}p^{-1} - (q(m,n,p))^{-1} &= p^{-1} - p'^{-1} - \theta(q(m,n))^{-1} \\ &= (p^{-1} - p'^{-1})(1 - (q(m,n))^{-1}) = (p^{-1} - p'^{-1})\tilde{q}^{-1}.\end{aligned}$$

(c) For $p \geq 2$, we use an adjointness argument to obtain

$$\|(\lambda - (iP(D) + V))^{-1}\|_{\mathcal{L}(L^p)} = \|(\overline{\lambda} - (i\overline{P}(D) + \overline{V}))^{-1}\|_{\mathcal{L}(L^{p'})}.$$

By the previous computation, this leads to the same estimate as in the case $p < 2$. But the conditions are to be written on p' in place of p: $p' > 2c/(c+1)$ is equivalent to $p < 2c/(c-1)$.

The conditions on r_1 and r_2 remain unchanged: they depend on $|p^{-1} - p'^{-1}|$.

The condition on c' is changed, but we only need $c' \geq 0$, which remains the case. Q.E.D.

REMARK. If $V_2 = 0$, then assumption (H3′) in Theorem 1 can be replaced by (H3).

REMARK. If $P(D)$ is homogeneous, then under the assumptions of Theorem 6, $(iP(D) + V)$ generates a smooth distribution group in $L^p(\mathbf{R}^n)$ of order any integer k with $k > n|1/p - 1/2| + 2$ and of exponential growth δ.

COROLLARY 1. *Assume P, V and p satisfy the assumptions of Theorem 6. Then there exists $\delta > 0$ such that for any λ with $|\operatorname{Re}\lambda| > \delta$ and any μ, $\operatorname{Re}\mu > 1$, we have the estimate*

$$\|(\lambda - (iP(D) + V))^{-\mu}\|_{\mathcal{L}(L^p(\mathbf{R}^n))} < C(\mu)|\operatorname{Re}\lambda - \varepsilon\delta|^{-\operatorname{Re}\mu}(|\lambda - \varepsilon\delta|\,|\operatorname{Re}\lambda - \varepsilon\delta|^{-1})^k,$$

where ε is the sign of $\operatorname{Re}\lambda$, k is any integer with $k > 2 + (n+3)|p^{-1} - 2^{-1}|$ and $C(\mu) = C\Gamma(\operatorname{Re}\mu)|\Gamma(\mu)|^{-1}|\mu|^k$.

PROOF. It is a consequence of Theorem 6 and Proposition 1.

COROLLARY 2. *Assume P, V and p satisfy the assumptions of Theorem 6. Let k be any integer with $k > (n+3)|p^{-1} - 2^{-1}| + 2$. Assume $U_0 \in L^p(\mathbf{R}^n)$ with $(iP(D) + V)^k U_0 \in L^p(\mathbf{R}^n)$. Then the solution $U(t, x)$ of the Cauchy problem $(*)$ with Cauchy data U_0 is a continuous function of the t-variable with values in $L^p(\mathbf{R}^n)$ and for any $t \in \mathbf{R}$ we have the estimate*

$$\|U(t, \cdot)\|_{L^p(\mathbf{R}^n)} \leq C(\|U_0\|_{L^p} + \|(iP(D) + V)^k U_0\|_{L^p})(1 + |t|^k)e^{\delta t}.$$

PROOF. It is a consequence of Theorem 6 and Proposition 3.

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